

A PROPERTY OF THE FROBENIUS MAP OF A POLYNOMIAL RING

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ABSTRACT. Let $R = k[x_1, \dots, x_n]$ be a ring of polynomials in a finite number of variables over a perfect field k of characteristic $p > 0$ and let $F : R \rightarrow R$ be the Frobenius map of R , i.e. $F(r) = r^p$. We explicitly describe an R -module isomorphism $\mathrm{Hom}_R(F_*(M), N) \cong \mathrm{Hom}_R(M, F^*(N))$ for all R -modules M and N . Some recent and potential applications are discussed.

1. INTRODUCTION

The main result of this paper is Theorem 3.3 which is a type of adjointness property for the Frobenius map of a polynomial ring over a perfect field. The interest in this fairly elementary result comes from its striking recent applications (see [5, Sections 5 and 6] and [6]) and also from the fact that despite extensive inquiries we have not been able to find it in the published literature.

Especially interesting is the application in [6] where a striking new result on local cohomology modules in characteristic $p > 0$ is deduced from Theorem 3.3. There is no doubt that the same result is true in characteristic 0, but the only currently known (to us) proof is in characteristic $p > 0$, based on Theorem 3.3. This has motivated what promises to be a very interesting search for some new technique to extend the result in [6] to characteristic 0.

We believe the results of this paper hold a potential for further applications and we discuss some of them in the last section.

2. PRELIMINARIES

Let R be a regular (but not necessarily local) UFD, let R_s and R_t be two copies of R (the subscripts stand for *source* and *target*) and let $F : R_s \rightarrow R_t$ be a finite ring homomorphism. Let

$$F_* : R_t\text{-mod} \rightarrow R_s\text{-mod}$$

The first and third authors gratefully acknowledge NSF support through grants DMS-0202176 and DMS-0701127. The second author is supported in part by a 2009 Spring/Summer Research Fellowship from Department of Mathematics, University of Michigan.

be the restriction of scalars functor (i.e. $F_*(M)$ for every R_t -module M is the additive group of M regarded as an R_s -module via F) and let

$$F^!, F^* : R_s\text{-mod} \rightarrow R_t\text{-mod}$$

be the functors defined by $F^!(N) = \text{Hom}_{R_s}(R_t, N)$ and $F^*(N) = R_t \otimes_{R_s} N$ for every R_s -module N .

$F^*(N)$ and $F^!(N)$, for an R_s -module N , have a structure of R_s -module via the natural R_s -action on R_t , while $F_*(M)$, for an R_t -module M , retains its old R_t -module structure (from before the restriction of scalars). Thus $F^*(N)$, $F^!(N)$, and $F_*(M)$ are both R_s - and R_t -modules.

It is well-known that F^* is left-adjoint to F_* , i.e. there is an isomorphism of R_t -modules

$$(1) \quad \begin{aligned} \text{Hom}_{R_t}(F^*(N), M) &\cong \text{Hom}_{R_s}(N, F_*(M)) \\ f &\mapsto (n \mapsto f(1 \otimes n)) \\ (r \otimes n \mapsto rf(n)) &\leftarrow f \end{aligned}$$

which is functorial in M and N (see, for example, [2, II.5, p.110]). This holds without any restrictions on R .

The main result of this paper is based on a different type of adjointness (see 8 below) which is certainly not well-known (we could not find a reference). Unlike (1) it is not quite canonical but depends on a choice of a certain isomorphism ϕ (described in (6) below). And it requires the conditions we imposed on R , namely, a regular UFD.

Since R_t , being regular, is locally Cohen-Macaulay, it follows from the Auslander-Buchsbaum theorem that the projective dimension of R_t as R_s -module is zero, i.e. R_t is projective, hence locally free, as an R_s -module. It is a standard fact that in this case $F^!$ is right adjoint to F_* , [2, Ch. 3, Exercise 6.10] i.e. for every R_t -module M and every R_s -module N there is an R_t -module isomorphism

$$(2) \quad \begin{aligned} \text{Hom}_{R_s}(F_*(M), N) &\cong \text{Hom}_{R_t}(M, F^!(N)) \\ (m \mapsto f(1)) &\leftarrow f \end{aligned}$$

where $f = f(m) : R_t \rightarrow N$. This isomorphism is functorial both in M and in N .

Consider the R_t -module $H \stackrel{\text{def}}{=} \text{Hom}_{R_s}(R_t, R_s)$. Since R_t is a locally free R_s -module of finite rank, it is a standard fact [2, Ch 2, Exercise 5.1(b)] that there is an isomorphism of functors

$$(3) \quad F^! \cong H \otimes_{R_s} -,$$

i.e. for every R_s -module N there is an R_t -module isomorphism

$$(4) \quad \begin{aligned} H \otimes_{R_s} N &\cong \text{Hom}_{R_s}(R_t, N) \\ h \otimes n &\mapsto (r \mapsto h(r)n) \end{aligned}$$

and this isomorphism is functorial in N . Replacing $F^!$ by $H \otimes_{R_s} -$ in (2) produces an R_t -module isomorphism

$$(5) \quad \text{Hom}_{R_s}(F_*(M), N) \cong \text{Hom}_{R_t}(M, H \otimes_{R_s} N).$$

which is functorial in M and N .

It follows from [3, Kor. 5.14] that locally H is the canonical module of R_t . In particular, the rank of H as R_t -module is 1 and therefore H is R_t -module isomorphic to some ideal I of R_t . According to [3, Kor. 6.13] the quotient R_t/I is locally Gorenstein of dimension $\dim R_t - 1$, hence I has pure height 1. Since R_t is a UFD, I is principal, i.e. there is an R_t -module isomorphism

$$(6) \quad \phi : R_t \rightarrow H.$$

Hence ϕ induces an isomorphism of functors

$$(7) \quad R_t \otimes_{R_s} - \xrightarrow{\phi \otimes \text{id}} H \otimes_{R_s} -.$$

Replacing $H \otimes_{R_s} -$ by $F^* = R_t \otimes_{R_s} -$ in (5) via (7) produces an R_t -module isomorphism

$$(8) \quad \text{Hom}_{R_s}(F_*(M), N) \cong \text{Hom}_{R_t}(M, F^*(N))$$

which is functorial in M and N .

While the isomorphisms (2), (3), (4) and (5) are canonical, the isomorphisms (6), (7) and (8) depend on a choice of ϕ . Every R_t -module isomorphism $\phi' : R_t \rightarrow H$ is obtained from a fixed ϕ by multiplication by an invertible element of R_t , i.e. $\phi' = c \cdot \phi$ where $c \in R_t$ is invertible. Therefore the isomorphisms (7) and (8) are defined up to multiplication by an invertible element of R_t .

Since the element $1 \in R_t$ generates the R_s -submodule $F(R_s)$ of R_t and does not belong to $\mathfrak{m}R_t$ for any maximal ideal \mathfrak{m} of R_s , the R_s -module $R_t/F(R_s)$ is projective. Hence applying the functor $\text{Hom}_{R_s}(-, R_s)$ to the injective map $F : R_s \rightarrow R_t$ produces a surjection $H \rightarrow \text{Hom}_{R_s}(R_s, R_s)$. Composing it with the standard R_s -module isomorphism $\text{Hom}_{R_s}(R_s, R_s) \xrightarrow{\psi \mapsto \psi(1)} R_s$ produces an R_s -module surjection $H \rightarrow R_s$. Composing this latter map with the isomorphism ϕ from (6) produces an R_s -module surjection

$$(9) \quad \psi : R_t \rightarrow R_s.$$

If N is an R_s -module, applying $- \otimes_{R_s} N$ to ψ produces an R_s -module surjection

$$(10) \quad \psi_N : R_t \otimes_{R_s} N \rightarrow N.$$

It is not hard to check that the isomorphism (8) sends $g \in \text{Hom}_{R_t}(M, F^*(N))$ to $\psi_N \circ g \in \text{Hom}_{R_s}(F_*(M), N)$.

3. THE MAIN RESULT

For the rest of this paper R is a ring of polynomials in a finite number of variables over a perfect field k of characteristic $p > 0$ and $F : R_s \rightarrow R_t$ is the standard Frobenius map, i.e. $F(r) = r^p$. The main result of this paper (Theorem 3.3) is an explicit description of the isomorphism (8) in terms of polynomial generators of R . The recent applications [5, 6] crucially depend on this explicit description. We keep the notation of the preceding section.

Let x_1, \dots, x_n be some polynomial generators of R over the field k , i.e. $R = k[x_1, \dots, x_n]$. We denote the multi-index i_1, \dots, i_n by \bar{i} . Since k is perfect, R_t is a free R_s -module on the p^n monomials $e_{\bar{i}} \stackrel{\text{def}}{=} x_1^{i_1} \cdots x_n^{i_n}$ where $0 \leq i_j < p$ for every j . If $i_j < p - 1$, then $x_j e_{\bar{i}} = e_{\bar{i}'}$ where \bar{i}' is the multi-index $i_1, \dots, i_{j-1}, i_j + 1, i_{j+1}, \dots, i_n$. If $i_j = p - 1$, then $x_j e_{\bar{i}} = x_j^p e_{\bar{i}'}$ where \bar{i}' is the multi-index $i_1, \dots, i_{j-1}, 0, i_{j+1}, \dots, i_n$.

Let $\{f_{\bar{i}} \in H \mid 0 \leq i_j < p \text{ for every } j\}$ be the dual basis of H , i.e. $f_{\bar{i}}(e_{\bar{j}}) = 1$ if $\bar{i} = \bar{j}$ and $f_{\bar{i}}(e_{\bar{j}}) = 0$ otherwise. If $i_j > 0$, then $x_j f_{\bar{i}} = f_{\bar{i}'}$ where \bar{i}' is the multi-index $i_1, \dots, i_{j-1}, i_j - 1, i_{j+1}, \dots, i_n$. If $i_j = 0$, then $x_j f_{\bar{i}} = x_j^p f_{\bar{i}'}$ where \bar{i}' is the multi-index $i_1, \dots, i_{j-1}, p - 1, i_{j+1}, \dots, i_n$.

Denote the multi-index $p - 1, \dots, p - 1$ by $\overline{p - 1}$ and let $\overline{p - 1} - \bar{i}$ be the multi-index $p - 1 - i_1, \dots, p - 1 - i_n$.

Proposition 3.1. (cf. [1, Remark 3.11]) *The R_s -linear isomorphism $\phi : R_t \rightarrow H$ that sends $e_{\bar{i}}$ to $f_{\overline{p - 1} - \bar{i}}$ is R_t -linear.*

Proof. All we have to show is that $\phi(x_j e_{\bar{i}}) = x_j \phi(e_{\bar{i}})$ for all indices j and multi-indices \bar{i} . This is straightforward from the definition of ϕ and the above description of the action of x_j on $e_{\bar{i}}$ and $f_{\bar{i}}$. \square

Clearly, $F^*(N) = R_t \otimes_{R_s} N = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} N)$, as R_s -modules. Thus every R_s -linear map $g : M \rightarrow F^*(N)$ has the form $g = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g_{\bar{i}})$ where $g_{\bar{i}} : M \rightarrow N$ are R_s -linear maps (i.e. $g_{\bar{i}} : F_*(M) \rightarrow N$ because M with its R_s -module structure is $F_*(M)$).

Lemma 3.2. *An R_s -linear map $g : M \rightarrow F^*(N)$ as above is R_t -linear if and only if $g_{\bar{i}}(-) = g_{\overline{p - 1} - \bar{i}}(e_{\overline{p - 1} - \bar{i}}(-))$ for every \bar{i} (here $e_{\overline{p - 1} - \bar{i}} \in R_t$ acts on $(-) \in F_*(M)$ via the R_t -module structure on $F_*(M)$).*

Proof. Assume g is R_t -linear. Then g commutes with multiplication by every element of R_t and in particular with multiplication by $e_{\overline{p - 1} - \bar{i}}$. That is $g(e_{\overline{p - 1} - \bar{i}}(-)) = e_{\overline{p - 1} - \bar{i}} g(-)$. Since $e_{\overline{p - 1} - \bar{i}} e_{\bar{i}} = e_{\overline{p - 1}}$, the $\overline{p - 1}$ -component of $e_{\overline{p - 1} - \bar{i}} g(-)$ is $e_{\overline{p - 1}} \otimes_{R_s} g_{\bar{i}}(-)$ while the $\overline{p - 1}$ -component of $g(e_{\overline{p - 1} - \bar{i}}(-))$ is $e_{\overline{p - 1}} \otimes_{R_s} g_{\overline{p - 1} - \bar{i}}(e_{\overline{p - 1} - \bar{i}}(-))$. Since the two $\overline{p - 1}$ -components are equal, $g_{\bar{i}}(-) = g_{\overline{p - 1} - \bar{i}}(e_{\overline{p - 1} - \bar{i}}(-))$.

Conversely, assume $g_{\bar{i}}(-) = g_{\overline{p - 1} - \bar{i}}(e_{\overline{p - 1} - \bar{i}}(-))$ for every \bar{i} . To show that g is R_t -linear all one has to show is that g commutes with the action of every $x_j \in R_t$, i.e. the \bar{i} -components of $g(x_j(-))$ and $x_j g(-)$ are the same for all \bar{i} . If $i_j > 0$, then the \bar{i} -component of $x_j g(-)$ is $e_{\bar{i}} \otimes_{R_s} g_{\bar{i}'}(-)$ where \bar{i}' is

the index $i_1, \dots, i_{j-1}, i_j - 1, i_{j+1}, \dots, i_n$ while the \bar{i} -component of $g(x_j(-))$ is $e_{\bar{i}} \otimes_{R_s} g_{\bar{i}}(x_j(-))$. But the fact that $g_{\bar{i}}(-) = g_{p-1}(\overline{e_{p-1-\bar{i}}(-)})$ for every \bar{i} implies $g_{\bar{i}'}(-) = g_{\bar{i}}(x_j(-))$.

If $i_j = 0$, then the \bar{i} -component of $x_j g(-)$ is $e_{\bar{i}''} \otimes_{R_s} ({}_s x_j g_{\bar{i}''}(-))$ where ${}_s x_j$ denotes the element of R_s corresponding to $x_j \in R_t$ (i.e. $F({}_s x_j) = x_j^p$) and \bar{i}'' is the index $i_1, \dots, i_{j-1}, p-1, i_{j+1}, \dots, i_n$ while the \bar{i} -component of $g(x_j(-))$ is $g_{\bar{i}}(x_j(-))$. But the fact that $g_{\bar{i}}(-) = g_{p-1}(\overline{e_{p-1-\bar{i}}(-)})$ for every \bar{i} implies ${}_s x_j g_{\bar{i}''}(-) = g_{\bar{i}}(x_j(-))$. \square

Finally we are ready for the main result of the paper which is the following explicit description of the isomorphism (8) for the Frobenius map.

Theorem 3.3. *For every R_t -module M and every R_s -module N there is an R_t -linear isomorphism*

$$\begin{aligned} \text{Hom}_{R_s}(F_*(M), N) &\cong \text{Hom}_{R_t}(M, F^*(N)) \\ (11) \quad g_{p-1}(-) &\leftarrow (g = \oplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g_{\bar{i}}(-))) \\ (12) \quad g &\mapsto \oplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g(\overline{e_{p-1-\bar{i}}(-)})). \end{aligned}$$

Proof. As is pointed out at the end of the preceding section, the isomorphism (8) sends $g \in \text{Hom}_{R_t}(M, F^*(N))$ to $\psi_N \circ g \in \text{Hom}_R(F_*(M), N)$. It is straightforward to check that with ϕ as in Proposition 3.1 the map ψ of (9) sends e_{p-1} to 1. This implies that $\psi_N \circ g = g_{p-1}$ and finishes the proof that formula (11) produces the isomorphism of (8) in one direction. The fact that the other direction of this isomorphism is according to formula (12) has essentially been proven in Lemma 3.2. \square

4. POTENTIAL APPLICATIONS

The notion of F -finite modules was introduced in [4]. An F -finite module is determined by a generating morphism, i.e. an R -module homomorphism $\beta : M \rightarrow F^*(M)$ where M is a finite R -module. For simplicity assume the R -module M has finite length, i.e. the dimension of M as a vector space over k , which we denote by d , is finite. Then the dimension of $F^*(M)$ as a vector space over k equals $p^n \cdot d$. The number p^n can be huge even for quite modest values of p and n . Thus the target of β may be a huge-dimensional vector space even if d, p and n are fairly small. But the R -module $F_*(M)$ has dimension d as a k -vector space, hence the map $\tilde{\beta} : F_*(M) \rightarrow M$ that corresponds to β under the isomorphism of Theorem 3.3, is a map between two d -dimensional vector spaces. Huge-dimensional vector spaces do not appear! This should make the map $\tilde{\beta}$ easier to manage computationally than the map β . Of course the isomorphism of Theorem 3.3 means that many properties of β could be detected in $\tilde{\beta}$; for example, β is the zero map if and only if $\tilde{\beta}$ is. Therein lies the potential for using Theorem 3.3 to make computations more manageable.

The functors $\text{Ext}_R^i(-, R)$ commute with both F^* and F_* . More precisely, for every finitely generated R_t -module M and every finitely generated R_s -module N there exist functorial R_t -module isomorphisms

$$\kappa_i : \text{Ext}_{R_t}^i(F^*(N), R_t) \cong F^*(\text{Ext}_{R_s}^i(N, R_s))$$

$$\lambda_i : F_*(\text{Ext}_{R_t}^i(M, R_t)) \cong \text{Ext}_{R_s}^i(F_*(M), R_s).$$

Indeed, for $i = 0$, $M = R_t$ and $N = R_s$ a straightforward composition of the R_t -module isomorphism $R_t \otimes_{R_s} R_s \xrightarrow{r' \otimes r \mapsto r' r^p} R_t$ and the standard R -module isomorphisms $\text{Hom}_R(R, R) \cong R$ for $R = R_t, R_s$ produce κ_0 while an additional R_t -module isomorphism $\text{Hom}_{R_s}(R_t, R_s) = H \cong R_t$ produces λ_0 . This implies that if $-$ stands for a complex of finite free R -modules, then $\text{Hom}_R(-, R)$ commutes with F^* and F_* . Taking now finite free resolutions of M and N in the categories of R_t - and R_s -modules respectively and considering that F^* and F_* , being exact, commute with the operation of taking the (co)homology of complexes, we get κ_i and λ_i for every i .

It is straightforward to check that there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{R_t}(F^*(N), M) & \longrightarrow & \text{Hom}_{R_s}(N, F_*(M)) \\ \downarrow & & \downarrow \\ \text{Hom}_{R_t}(\text{Ext}_{R_t}^i(M, R_t), \text{Ext}_{R_t}^i(F^*(N), R_t)) & & \text{Hom}_{R_s}(\text{Ext}_{R_s}^i(F_*(M), R_s), \text{Ext}_{R_s}^i(N, R_s)) \\ \downarrow & & \downarrow \\ \text{Hom}_{R_t}(\text{Ext}_{R_t}^i(M, R_t), F^*(\text{Ext}_{R_s}^i(N, R_s))) & \longrightarrow & \text{Hom}_{R_s}(F_*(\text{Ext}_{R_t}^i(M, R_t)), \text{Ext}_{R_s}^i(N, R_s)) \end{array}$$

where the top horizontal map is the isomorphism (1), the bottom horizontal map is the isomorphism of Theorem 3.3, the bottom vertical maps are induced by κ_i and λ_i and, finally, the top vertical maps are defined by sending every $f \in \text{Hom}(L, L')$ to the map $\text{Ext}^i(L', R) \rightarrow \text{Ext}^i(L, R)$ functorially induced by f .

In other words, if a pair of maps $F^*(N) \rightarrow M$ and $N \rightarrow F_*(M)$ correspond to each other under (1), then the induced maps $\text{Ext}_{R_t}^i(M, R_t) \rightarrow F^*(\text{Ext}_{R_s}^i(N, R_s))$ and $F_*(\text{Ext}_{R_t}^i(M, R_t)) \rightarrow \text{Ext}_{R_s}^i(N, R_s)$ correspond to each other under isomorphism of Theorem 3.3.

Of course there is a similar diagram with the isomorphism of Theorem 3.3 in the top row; we are not going to use it in the rest of the paper.

An important example of F -finite modules are local cohomology modules $H_I^i(R)$ of R with support in an ideal $I \subset R$. A generating morphism of $H_I^i(R)$ is the composition

$$f : \text{Ext}_R^i(R/I, R) \rightarrow \text{Ext}_R^i(F^*(R/I), R) \xrightarrow{\kappa_i} F^*(\text{Ext}_R^i(R/I, R))$$

where the first map is induced by the isomorphism $F^*(R/I) \xrightarrow{r' \otimes \bar{r} \mapsto r' \bar{r}^p} R/I^{[p]}$ followed by the natural surjection $R/I^{[p]} \rightarrow R/I$. The following proposition

holds the potential for simplifying computations involving local cohomology modules.

Proposition 4.1. *Let $I \subset R$ be an ideal and let the composition*

$$f : \operatorname{Ext}_R^i(R/I, R) \rightarrow \operatorname{Ext}_R^i(F^*(R/I), R) \xrightarrow{\kappa_i} F^*(\operatorname{Ext}_R^i(R/I, R))$$

be as above. The map that corresponds to f under the isomorphism of Theorem 3.3 is the composition

$$g : F_*(\operatorname{Ext}_R^i(R/I, R)) \xrightarrow{\lambda_i} \operatorname{Ext}_R^i(F_*(R/I), R) \rightarrow \operatorname{Ext}_R^i(R/I, R)$$

where the second map in the composition is nothing but the map induced on $\operatorname{Ext}_R^i(-, R)$ by the natural Frobenius map $R/I \xrightarrow{r \mapsto r^p} F_(R/I)$.*

Proof. This is immediate from the above commutative diagram considering that the maps $F^*(R/I) \xrightarrow{r' \otimes r \mapsto r' r^p} R/I$ and $R/I \xrightarrow{r \mapsto r^p} F_*(R/I)$ correspond to each other under the isomorphism (1). \square

In addition to potential use in computation, the material of this section has already been used in a proof of a theoretical result [5, Section 5].

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